Discussion of:

Which (Nonlinear) Factor Models?

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In a nutshell: method

Three key ingredients:

1. **SDF recovery**

   Minimum divergence between $\mathbb{Q}$ and $\mathbb{P}$ under Euler equation constraint:

$$\widehat{\mathbb{Q}} = \arg\min \; D(\mathbb{Q}||\mathbb{P};\gamma) \; s.t. \; \mathbf{0} = \mathbb{E}^{\mathbb{P}}\left[M\mathbf{R^e}\right] = \mathbb{E}^{\mathbb{Q}}\left[\mathbf{R^e}\right] \;\; \Rightarrow \;\; \widehat{M}\left(\mathbf{R^e}\right) \propto \widehat{\frac{d\mathbb{Q}}{d\mathbb{P}}}$$

   (e.g., Stutzer, 1995, Ghosh-Julliard-Taylor, 2016, 2019, Almeida-Garcia, 2017, Ghosh-Julliard-Stutzer, 2020)

2. **Constraint on SDF components – aka, the "models"**

   Low dimensional tradable factors ($\mathbf{F}_j$) models in the literature, i.e. as above but

$$\mathbf{0} = \mathbb{E}^{\mathbb{P}}\left[M\mathbf{F}_j\right] = \mathbb{E}^{\mathbb{Q}}\left[\mathbf{F}_j\right] \;\; \Rightarrow \;\; \widehat{M}\left(\mathbf{F}_j\right) \propto \widehat{\frac{d\mathbb{Q}}{d\mathbb{P}}}$$

3. **SDF Sharpe ratio comparisons**

   SR^2 of $\widehat{M}\left(\mathbf{F}_j\right)$:= SR^2 of test assets $-$ SR^2 of alphas from GLS CSR on $\widehat{M}\left(\mathbf{F}_j\right)$.

Note: test assets matter (cf. Barillas-Shanken, 2017)

In a nutshell: findings

For some $\gamma > 0$, SR^2 of $\widehat{M}\left(\mathbf{F}_j\right) >> SR^2$ of $\mathbf{F}_j$.

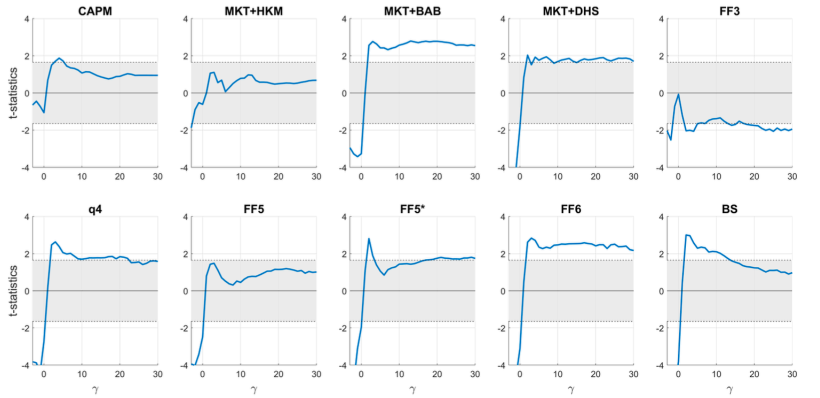


Fig. 4: **Statistical significance of Sharpe ratio difference across γ.** This figure plots, for each factor model f and each γ, the t-statistics for the difference between $Sh^2(m_{\gamma,p})$ and $Sh^2(m_p^*) = Sh^2(f)$. The t-statistics is derived using the asymptotic test of Barillas et al. (2020). We consider $\gamma \in [-3, 30]$, with a grid with spacing of 1. The gray area denotes the region of statistical insignificance at the 10% level. The sample ranges from July, 1972 to October, 2018.

Which divergence? (Which γ?)

- Overwhelmingly, the previous literature has focused on relative entropy, i.e. just two cases:

$\gamma = -1$: akin to Owen's Empirical Likelihood

$\gamma = 0$: akin to Kitamura's Exponential Tilting

Why? 1) MLE analogous; 2) minimum KLIC; 3) appropriate for tail risk/large deviations; 4) Bayesian interpretation (approximated/exact)

Note: in the data, SDF estimates with the above tend to be almost identical

- This paper uses the Cressie-Read (1984) family as Almeida-Garcia (2017)

$\Rightarrow$ "free" γ ($\in [-3, 30]$), micro-founded as HARA preferences SDF...
... but not all HARA preferences are born equal...
... and they come with constraints!

Which γ? cont'd

Recall: HARA $\Leftrightarrow$ 1/ARA is linear in final wealth/state variable

If $\gamma > 0$:

1. ARA $\uparrow$ as wealth/state $\uparrow$

$\Rightarrow$ the richer you are the less you want to invest in risky assets.

2. Utility function well behaved up to an upper bound on wealth/state variable.

Example: satiation point of quadratic utility, after which canonical TVC does not hold and optimal policy not pinned down by Euler equation as usual.

3. Ignoring the bound, $\mathbb{Q}$ and $\mathbb{P}$ are not absolutely continuous if the upper bound is crossed with non-zero probability (different zero prob. sets)

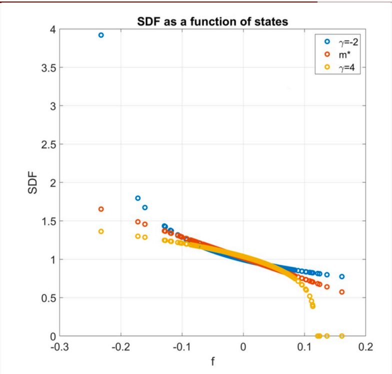
$\Rightarrow$ marginal utility stays at zero from upper bound onward, and an assets that pays only in these states has a price of zero.

¡ SDF recovery breaks down! (measures are not equivalent)

Note: $\gamma < 0$ gives a lower bound, but that is less problematic: e.g. consumption can't be negative / no-Ponzi-game condition (and dual form with $\gamma = -1$ ensures no violation, in sample, of absolute continuity).

Is $\gamma > 0$ a problem in the applications?

Yes for the CAPM!



... and exactly at the values for which SR^2 of $\widehat{M}\left(R^{MKT}\right) >> SR^2$ of R^{MKT}.

$\Rightarrow$ Must check for all other models!

Suggestions: 1) I'd drop the HARA interpretation (or do it properly with domain constraint) and focus on statistical properties; 2) cross-validate to choose one γ; 3) regularise the problem to ensure absolute continuity of $\mathbb{Q}$ and $\mathbb{P}$

Fixing the null and testing it formally

- The paper compares linear SDFs to non-linear minimum discrepancy ones, but you could actually fix the “null” and test wether the non-linearities are need.

How? Ghosh-Julliard-Taylor (2016) considers SDFs of the form (with $\gamma = 0$ or -1)

$$M_t = \underbrace{m(X_t)}_{observable} \times \underbrace{\psi_t}_{latent}$$

$$\widehat{\Psi} = \arg\min D(\Psi||\mathbb{P};\gamma) \; s.t. \; \mathbf{0} = \mathbb{E}^{\mathbb{P}}\left[M\mathbf{R^e}\right] = \mathbb{E}^{\Psi}\left[m(X)\mathbf{R^e}\right] \;\Rightarrow\; \widehat{\psi}\left(m(X)\mathbf{R^e}\right) \propto \widehat{\frac{d\Psi}{d\mathbb{P}}}$$

- easy to extend to other values of γ (usual dual solution, on the valid domain, but with $m(X_t)$ scaling returns).

$\Rightarrow$ set $m(X_t)$ to be the linear SDF and:

1. can test formally if non-linearities are needed (χ^2 test)
2. can decompose how much of the pricing ability comes from each component, globally and on a cumulant-by-cumulant basis (exact result for $\gamma = -1$ or 0)

Misspecification, bias & sparsity

1. Ghosh-Otsu (2021) implies that SDF recovery (with $\gamma = 0$ or -1) is biased under mispecification

$\Rightarrow$ result likely holds for other γ (and CV would likely fail, Sueishi, 2017)

2. The "true" latent (linear) SDF is dense in the space of observable factors (Bryzgalova-Huang-Julliard, 2023, (equity) & Dickerson-Julliard-Mueller, 2023, (bonds)).

$\Rightarrow$ Low-dimensional linear SDFs the paper starts from are all misspecified

1+2= Need to tackle the bias/omitted variables formally (e.g., Giglio-Xiu, 2021, or at least acknowledge it)

But: sparsity is not invariant to transformations (Giannone-Lenza-Primiceri, 2021)

Q: Do non-linearities replace the need for a dense SDF? Can be tested formally! (as a Bayesian at least)

Conclusion & Final Suggestions

Baseline: A clever, interesting, and well written, paper, that makes important points, and that I enjoyed reading.
$\Rightarrow$ a lot of upside potential

But I would not submit it yet:

- Absolute continuity of $\mathbb{Q}$ and $\mathbb{P}$ with $\gamma > 0$ needs fixing
- Would benefit from going back to what the non-linearities capture exactly (note: link of dual Lagrange multipliers and portfolio weights of HARA agent)
- Proposed extension to nontradable factors will not work for weak ones
- The bootstrap procedure needs to be fully spelled out

... and I fully agree with Olivier's suggestions – take them on board!